Two-Qubit Quantum Circuits
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Outline
- Introduction
- Background
- Worst-case Optimal Two-Qubit Circuits
- CNOT Counting
- Synthesis in the Context of Measurement
- Open Questions

Motivation for Two Qubit Synthesis
- Many implementation technologies limited
  - By two or three qubits
- 2-3 qubits enough for q. communication
- Peephole circuit optimization
- Two qubit problems are easy!

Before We Looked At This Problem
- Algorithms known for n-qubit synthesis
  - Barenco et Al.
  - Cybenko
  - Tucci (?)
  - Many others
- Worst-case performance
  - $O(n 4^n)$ basic gates
  - 61 elementary gates for 2-qubits
Our Methods, Worst Case

<table>
<thead>
<tr>
<th>Gate Library</th>
<th># CNOT gates</th>
<th># 1-qubit gates</th>
<th>Total # gates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_z, R_y, CNOT$</td>
<td>3</td>
<td>15</td>
<td>18</td>
</tr>
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<td>3</td>
<td>15</td>
<td>18</td>
</tr>
<tr>
<td>Basic Gates (1-qubit gates, CNOT)</td>
<td>3</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

- Except for the red 7 and 10, can’t do better

Our Methods, Any Case

- Yield circuits with optimal CNOT count
  - Recall: CNOT gates expensive in practice
- However, may have excess 1-qubit gates

Our Methods, Specific Cases

- Automatically detect tensor products
  - Tensor-product circuits require no CNOT gates
- Yield an optimal circuit for 2-qubit QFT
  - 6 basic gates (3 CNOT gates)

What We Do Differently

- Emphasize circuit identities
- Think in elementary (not basic) gates
- Compute circuit-structure invariants
- Avoid cumbersome physics notation
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Circuit Identities

- Used to cancel, combine, and rearrange gates in a circuit

For example:

```
\[ \begin{array}{c}
\text{CNOT} \\
\text{SWAP}
\end{array} \]
```

Circuit Identities

<table>
<thead>
<tr>
<th>Circuit identities</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ C</td>
<td>C ] = 1</td>
</tr>
<tr>
<td>[ x ] [ x ] = 1</td>
<td>SWAP-gate cancellation</td>
</tr>
<tr>
<td>[ x ] [ x ] = 1</td>
<td>CNOT-gate elimination</td>
</tr>
<tr>
<td>[ C</td>
<td>H</td>
</tr>
<tr>
<td>[ C</td>
<td>S</td>
</tr>
<tr>
<td>[ C</td>
<td>Z ] = [ Z</td>
</tr>
<tr>
<td>[ x^{-1}</td>
<td>x^{-1} ] = 1</td>
</tr>
<tr>
<td>[ R_y (\theta) ] [ R_y (\theta) ] = [ R_y (\theta + \phi) ]</td>
<td>Merging R_y gates</td>
</tr>
<tr>
<td>[ \phi ] [ \phi ] = [ R_x (\phi) ] [ R_x (\phi) ]</td>
<td>Changing axis of rotation</td>
</tr>
<tr>
<td>[ \text{H}</td>
<td>H ] [ H</td>
</tr>
<tr>
<td>[ \sigma_x</td>
<td>\sigma_x ] = [ C ]</td>
</tr>
<tr>
<td>[ \theta</td>
<td>\theta ] = [ C ]</td>
</tr>
</tbody>
</table>

Elementary vs. Basic Gates

- The basic gate library
  - One-qubit operators, plus the CNOT
  - Is universal

- Many gate-counts given in basic gates
  - Eg., those in Barenco et. Al.
Elementary vs. Basic Gates

- A “smaller” gate library will suffice
  - Express one-qubit operators as a product
    - $e^{i\varphi}R_z(a) \cdot R_y(b) \cdot R_z(c)$
  - The elementary-gate library
    - $\{\text{CNOT, } R_y, R_z\}$
    - Parameterized gates $R_y, R_z$
      - One dimensional

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The Canonical Decomposition

- Any $U$ in $U(4)$ can be written as
  \[ U = e^{i\varphi} [a \otimes b] \delta [f \otimes g] \]
  - where $a, b, f, g$ are in $SU(2)$
  - And $\delta$ is diagonal in the magic basis

The Magic Basis

- Given by the rows of
  \[ E = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix} \]
  - In the magic basis
    - $SO(4)$ operators look like tensor products
The 18 Gate Construction

- Given $U$ in SU(4)
- Use the canonical decomposition
  \[ e^{i\pi/4} \chi^{1,2} U = \begin{bmatrix} a \otimes b \end{bmatrix} \delta \begin{bmatrix} f \otimes g \end{bmatrix} \]
- $a, b, f, g$ are in SU(2)
- $\delta$ is diagonal in the magic basis

The 18 Gate Construction

- Change basis: $\delta = E \Delta E^*$
  - $\Delta$ diagonal in the computational basis
- Implement $E, \Delta$

\[
E = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\Delta = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0
\end{bmatrix}
\]

The 18 Gate Construction

- Concatenate circuits & apply identities
  - The one-qubit gates associated with $E$ vanish
  - Since we began with $e^{i\pi/4} \chi^{1,2} U$, can remove a CNOT
- The resulting circuit requires
  - 3 CNOT gates and 15 $R_x / R_y$ gates
  - Or, 3 CNOT gates and 7 one-qubit gates

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FIG. 1: Our generic circuit with three CNOT gates that can implement an arbitrary two-qubit operator. It requires ten basic gates [2] or eighteen gates from the library (CNOT, $R_x$, $R_y$).
Other Gate Libraries

- One can use $R_x$ instead of $R_z$
  - Proof: Conjugate by Hadamard
- However, no worst-case optimal circuit using $R_x$, $R_z$, CNOT “looks like” this circuit
  - Because both $R_x$, $R_z$ can pass through CNOT

Worst-case Optimality

- Proven by dimension counting
  - The dimension of SU(4) is 15
  - Need 15 parameterized gates
  - Need 3 CNOT gates to prevent cancellations

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Counting CNOT Gates

- Define $\gamma(U) = U [\sigma^y \otimes \sigma^y] U^\dagger [\sigma^y \otimes \sigma^y]$
- Observe that for $u$ a 2 by 2 matrix, $u \sigma^y u^\dagger \sigma^y = I \cdot \det(u)$
- It follows that
  - If $U = (a \otimes b) V (c \otimes d)$
  - $\gamma(U)$ is similar to $\gamma(V)$
- Thus the conjugacy class (or characteristic polynomial) of $\gamma(U)$ is constant on the equivalence classes of two-qubit computations up to one-qubit gates
Counting CNOT Gates

- Theorem: if $U$ requires at least
  - 3 CNOT gates, $\text{tr} \gamma(U)$ is not real
  - 2 CNOT gates, $\gamma(U) \neq I$ and $\gamma(U)^2 \neq I$
  - 1 CNOT gate, $\gamma(U) \neq I$

- Only depends on conjugacy class of $\gamma(U)$

Counting CNOT Gates

- The proof builds a CNOT-optimal circuit computing the desired operator

- Fully constructive & soon to be implemented and made web-accessible

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Measurement

- Measurement kills phase

- The following concepts are the same
  - Knowing that you will measure
    - In the computational basis
  - Having a synthesis don’t care
    - Left multiplication by a diagonal operator $\Delta$
CNOT Counting + Measurement

- To count the number of CNOTs needed to compute U in the context of measurement
  - Apply the CNOT counting theorem to all possible matrices ΔU
- Observation: γ(A'B) = γ(A)γ(B)

CNOT Counting + Measurement

- In fact, 2 CNOTs and 12 elementary gates suffice to simulate an arbitrary 2-qubit operator, up to measurement
  - Dimension arguments show this is optimal
- It follows that 18 gates from \{R_x, R_z, CNOT\} suffice to compute any 2-qubit operator
  - I can’t show this without the CNOT counting formula

CNOT Counting + Measurement

- Other measurements are possible
  - (2+2) measurements
  - (3+1) measurements
- Can characterize CNOT-optimal circuits
  - If subspaces spanned by comp. basis vectors

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Open Questions

- It has been asserted that 6 two-qubit gates suffice to compute any 3 qubit gate.
- If this is true, we can give a circuit for an arbitrary three-qubit operator.
  - Worst-case suboptimal by only 4 CNOT gates.
- But, no proof of this assertion in the literature,
  - Numerical evidence supporting it is weak.
- Is this assertion true?

Open Questions

- A recent paper gives worst-case optimal 2-qubit circuits for arbitrary c-U gates.
  - Generalizes our 18-gate construction.
- Can the CNOT-counting formula be similarly generalized?

Open Questions

- We have CNOT-optimal circuits given measurement in the computational basis.
- It is harder to use the counting formula for measurements in other bases.
- What happens here?
- In particular:
  - Is there any basis B in which making a (3+1) measurement

Thank You For Your Attention