Automorphisms of graphs

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Abstract

This chapter surveys automorphisms of finite graphs, concentrating on the asymmetry of typical graphs, prescribing automorphism groups (as either permutation groups or abstract groups), and special properties of vertex-transitive graphs and related classes. There are short digressions on infinite graphs and graph homomorphisms.

1 Graph automorphisms

An *automorphism* of a graph G is a permutation g of the vertex set of G with the property that, for any vertices u and v, we have $ug \sim vg$ if and only if $u \sim v$. (As usual, we use vg to denote the image of the vertex v under the permutation g. See [13] for the terminology and main results of permutation group theory.)

This simple definition does not suffice for multigraphs; we need to specify a permutation of the edges as well as a permutation of the vertices, to ensure that the multiplicity of edges between two vertices is preserved. (Alternatively, a multigraph can be regarded as a *weighted graph*, where the weight $a_{u,v}$ is the number of edges from u to v; an automorphism is required to satisfy $a_{ug,vg} = a_{u,v}$. This gives a slightly different description of automorphisms, but the action on the set of vertices is the same.) We will consider only simple graphs here.

The set of all automorphisms of a graph G, with the operation of composition of permutations, is a *permutation group* on VG (a subgroup of the symmetric group on VG). This is the *automorphism group* of G, denoted $\mathrm{Aut}(G)$. We describe any subgroup $\mathcal{H}$ of $\mathrm{Aut}(G)$ as a *group of automorphisms* of G, and refer to $\mathrm{Aut}(G)$ as the *full automorphism group*.

More generally, an *isomorphism* from a graph G to a graph H is a bijection f from the vertex set of G to that of H such that $uf \sim wf$ (in H) if and only if $u \sim v$ (in G). We say that G and H are *isomorphic* (written $G \cong H$) if there is an isomorphism between them.

Among its other jobs, the automorphism group arises in the enumeration of graphs, specifically in the relation between counting labelled and unlabelled graphs. A *labelled graph* on n vertices is a graph whose vertex set is $\{1, \ldots, n\}$, while an *unlabelled graph* is simply an isomorphism class of n-element graphs. Now the number of labellings of a given unlabelled graph G on n vertices is $n!/|\mathrm{Aut}(G)|$. For a labelling is given by a bijective function F from $\{1, \ldots, n\}$ to VG; there are $n!$ such functions, and two of them (say F_1 and F_2) define the same labelled graph if and only if there is an automorphism g such that $F_2(i) = F_1(i)g$ for all $i \in \{1, \ldots, n\}$. Figure 1 shows the three labellings of the path of length 2 (a graph whose automorphism group has order 2).

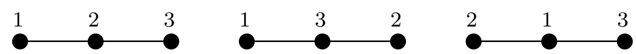

Figure 1: Labellings

The automorphism group is an algebraic invariant of a graph. Here are some simple properties. First, some notation:

- The *direct product* $\mathcal{G}_1 \times \mathcal{G}_2$ of two permutation groups $\mathcal{G}_1$ and $\mathcal{G}_2$ (acting on sets Ω_1 and Ω_2) is the permutation group on the disjoint union $\Omega_1 \cup \Omega_2$ whose elements are ordered pairs (g_1, g_2) for $g_i \in \mathcal{G}_i$; the action is given by
$$v(g_1, g_2) = \begin{cases} vg_1 & \text{if } v \in \Omega_1, \\ vg_2 & \text{if } v \in \Omega_2. \end{cases}$$
This notion extends to the direct product of any number of permutation groups.

- If $\mathcal{G}_2$ is a permutation group on $\{1, \ldots, n\}$, then the *wreath product* $\mathcal{G}_1 \wr \mathcal{G}_2$ is generated by the direct product of n copies of $\mathcal{G}_1$, together with the elements of $\mathcal{G}_2$ acting on these n copies of $\mathcal{G}_1$.
- Finally, $\mathcal{S}_n$ is the symmetric group on $\{1, \ldots, n\}$.

Theorem 1.1 *(a) A graph and its complement have the same automorphism group.*

(b) Let the connected components of G consist of n_1 copies of G_1, ..., n_r copies of G_r, where $G_1, \ldots, G_r$ are pairwise non-isomorphic. Then

$$\mathrm{Aut}(G) = (\mathrm{Aut}(G_1) \wr \mathcal{S}_{n_1}) \times \cdots \times (\mathrm{Aut}(G_r) \wr \mathcal{S}_{n_r}).$$

(c) $\mathrm{Aut}(K_n) = \mathcal{S}_n$.

In view of these results, we can reduce questions about automorphism groups to the case when the graphs are connected.

A recent survey with somewhat different emphasis is that of Babai and Goodman [5]. In addition, no serious student should be without the book [28], which contains surveys of a number of aspects of graph symmetry.

2 Algorithmic aspects

Two algorithmic questions arising from the above definitions are *graph isomorphism* and *finding the automorphism group*. The first is a decision problem:

Graph Isomorphism

Instance: Graphs G and H.

Question: Is $G \cong H$?

The second problem requires output. Note that a subgroup of $\mathcal{S}_n$ may be superexponentially large in terms of n; but any subgroup has a generating set of size $O(n)$, which specifies it in polynomial space.

Automorphism group

Instance: A graph G.

Output: Generating permutations for $\mathrm{Aut}(G)$.

The two problems are closely related. Indeed, the first has a polynomial reduction to the second. For suppose that we are given two graphs G and H. By complementing if necessary, we may assume that both G and H are connected. Now suppose that we can find generating permutations for $\mathrm{Aut}(K)$, where K is the disjoint union of G and H. Then G and H are isomorphic if and only if some generator interchanges the two connected components.

Conversely, if we can solve the graph isomorphism problem, we can at least check whether a graph has a non-trivial automorphism, by attaching distinctive "gadgets" at each of its vertices and checking whether any pair of the resulting graphs are isomorphic. (Finding generators for the automorphism group may be harder.)

The exact status of these two problems is unresolved. They belong to a select group of problems which belong to NP but are not known either to belong to P nor to be NP-complete. For some particular classes of graphs, notably graphs of bounded valency [43] and graphs with bounded eigenvalue multiplicity [7], the isomorphism problem is known to be polynomial. See [23] for the fundamentals of computational complexity.)

In practice, these questions can be resolved for graphs with thousands of vertices. Chapter **??** gives an account of the algorithms used and their implementation.

It turns out that, for almost all graphs, the algorithmic questions can be answered very quickly. However, "almost all" does not include some of the most interesting graphs, including strongly regular graphs (discussed in Chapter **??**).

3 Automorphisms of typical graphs

The smallest graph (apart from the one-vertex graph) whose automorphism group is trivial is shown in Figure 2.

However, small graphs are (as usual) not a reliable guide here. Erdős and Rényi [16] showed:

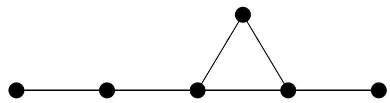

Figure 2: An asymmetric graph

Theorem 3.1 *Almost all graphs have no non-trivial automorphisms.*

That is, the proportion of graphs on n vertices which have a non-trivial automorphism tends to zero as $n \to \infty$. This is true whether we take labelled or unlabelled graphs. As noted in the introduction, the theorem implies that almost all graphs can be labelled in $n!$ different ways, so that the number of unlabelled graphs on n vertices is asymptotically $2^{n(n-1)/2}/n!$. (There are clearly $2^{n(n-1)/2}$ labelled graphs on the vertex set $\{1, \ldots, n\}$, since we can choose whether or not to join each pair of vertices by an edge.) There are now good estimates for the error term in the asymptotic expansion; it arises from graphs with non-trivial symmetry, and so these estimates quantify the theorem.

In fact, more is true. There are various methods for *canonical labelling* of a graph (for example, choosing the lexicographically least labelled graph in the isomorphism class). For almost all graphs, the canonical labelling is unique, and can be found in polynomial time; for such graphs, we can verify efficiently that their automorphism groups are trivial. Typically, graphs with regularity properties, such as strongly regular graphs (Chapter **??**), are hard for canonical labelling algorithms, even when their automorphism groups are trivial.

The theorem remains true for various special classes of graphs including regular graphs of fixed valency $k > 2$ (we can even allow the valency to grow, not too rapidly, with n: see [59]), or the prolific strongly regular graphs of Latin square or Steiner triple system type discussed in Chapter **??** (this uses the fact that almost all Latin squares or Steiner triple systems have no non-trivial automorphisms).

Other methods of quantifying the theorem can be found. For example, given any graph, we can alter it so that some two vertices have the same neighbour sets by altering at most $n/2$ adjacencies. The resulting graph has an automorphism interchanging the two vertices and fixing all others.

Erdős and Rényi [16] showed that, for almost all graphs, this is the "shortest distance to symmetry".

4 Permutation groups

The question, "Which permutation groups are the full automorphism groups of graphs?", has no easy answer. Given a permutation group $\mathcal{G}$ on a set Ω, we can describe all the graphs on which $\mathcal{G}$ acts as follows. There is a coordinatewise action of $\mathcal{G}$ on $\Omega \times \Omega$, given by $(u, v)g = (ug, vg)$. Let U be the set of all orbits of $\mathcal{G}$ on $\Omega \times \Omega$ which consist of pairs of *distinct* elements. There is a natural *pairing* of orbits in U, where the orbit paired with O is $O^* = \{(v, u) : (u, v) \in O\}$. Now let S be any subset of U which contains the orbit paired with each of its members, and define a graph $G(S)$ on the vertex set Ω by the rule that $u \sim v$ in $G(S)$ if and only if $(u, v) \in O$ for some $O \in S$. Then $G(S)$ is a simple graph admitting $\mathcal{G}$ as a group of automorphisms; and every such graph arises by this construction.

The construction easily adapts to other classes of graphs. For directed graphs, drop the requirement that S is closed under pairing; for graphs with loops, include all orbits on $\Omega \times \Omega$, not just those consisting of pairs of distinct elements; and for multigraphs, allow multisets of orbits. It is also very practical: instead of having to list all the edges in order to specify a graph, we need only give a set of orbit representatives. Chapter **??** describes how to exploit this observation.

A permutation group $\mathcal{G}$ on Ω is said to be 2-*closed* if every permutation of Ω which preserves all the $\mathcal{G}$-orbits in $\Omega \times \Omega$ belongs to $\mathcal{G}$. More generally, the 2-*closure* of $\mathcal{G}$ is the group of permutations which preserve all the $\mathcal{G}$-orbits. These concepts was introduced by Wielandt [56].

Now, for any graph G, the group $\mathrm{Aut}(G)$ is 2-closed; for the edge set of G is a union of orbits of $\mathrm{Aut}(G)$, and so is preserved by its 2-closure. The converse fails, but not too badly: in fact, a permutation group $\mathcal{G}$ is 2-closed if and only if it is the full automorphism group of an edge-coloured directed graph. (We associate a colour with each $\mathcal{G}$-orbit on pairs.)

However, it is not easy to decide if a permutation group is 2-closed.

The construction of a graph from a permutation group has been reversed; many of the sporadic simple groups discovered in the mid-twentieth century were constructed as groups of automorphisms of particular graphs. The simplest such construction is that of Higman and Sims [36]: the vertex set consists of a special symbol ∞ together with the 22 points and 77 blocks of the so-called *Witt design* $\mathcal{W}_{22}$. We join ∞ to all the points; we join a point and a block if and only if they are incident; and we join two blocks if and only if they are disjoint. It is clear that the automorphism group of $\mathcal{W}_{22}$ acts as a group of automorphisms of the graph, fixing ∞. It is not too hard to show that the full automorphism group is transitive; it contains the Higman–Sims simple group as a subgroup of index 2.

In the case of the Fischer groups [18], the group and the graph are even more closely related. The vertices of the graph are the elements of a particular conjugacy class of involutions in the group, two vertices being joined if and only if the involutions don't commute.

5 Abstract groups

If we consider groups as abstract algebraic objects rather than as concrete permutation groups, a clear-cut result is possible. Frucht [20] proved:

Theorem 5.1 *Every group is the automorphism group of a graph. If the group is finite, the graph can be taken to be finite.*

Subsequently, Frucht [21] showed that *every group is the automorphism group of a trivalent graph.* This has inspired a large number of similar results. We call a class $\mathcal{C}$ of structures *universal* if every finite group is the automorphism group of a structure in $\mathcal{C}$. The combined results of Frucht, Sabidussi, Mendelsohn, Babai, Kantor, and others show that the following types of graph or other structures are universal: graphs of valency k for any fixed $k > 2$; bipartite graphs; strongly regular graphs; Hamiltonian graphs; k-connected graphs, for $k > 0$; k-chromatic graphs, for $k > 1$; switching classes of graphs; lattices; projective planes (possibly infinite); Steiner triple systems; symmetric designs (BIBDs).

Here, for example, the universality of Steiner triple systems is shown by using a graph to construct an STS with the same symmetry; then the universality of strongly regular graphs is shown by taking the line graphs of Steiner triple systems (see Chapter **??**). The structures are finite in all cases except projective planes. (Whether every finite group is even a subgroup of the automorphism group of a finite projective plane is an open problem.)

Another class of results involves structures for which there is some obvious restriction on the automorphism group. For example, a tournament cannot admit an automorphism of order 2 (such an automorphism would necessarily interchange the ends of some arc) and so the automorphism group has odd order. This is the only restriction; Moon [46] showed that every group of odd order is the automorphism group of a tournament.

Pólya observed that not every group is the automorphism group of a tree: More precisely, the class of automorphism groups of trees is the smallest class containing the trivial group and closed under direct product and the operations 'wreath product with the symmetric group S_n of degree n' for each $n > 1$.

Automorphism groups of trees are of further importance in group theory. Any finite tree has either a vertex or an edge which is fixed by all automorphisms, according as it is central or bicentral. Things are very different for infinite trees. We say that a group $\mathcal{G}$ acts *freely* on a tree T if no non-identity element of $\mathcal{G}$ fixes any vertex or edge of T. Now Serre [49] showed:

Theorem 5.2 *A group is free if and only if it acts freely on a tree.*

This simple observation has led to the so-called *Bass–Serre theory* in combinatorial group theory, describing certain group constructions (amalgamated free products and HNN-extensions) in terms of their actions on a tree. See Dicks and Dunwoody [15] for an account of this.

Usually such a precise description does not exist. For example, every composition factor of the automorphism group of a planar graph is either a cyclic group or an alternating group. More generally, Babai showed:

Theorem 5.3 *If $\mathcal{G}$ is the automorphism group of a graph embeddable in the orientable surface of genus k, then there is a number $f(k)$ such that any composition factor of $\mathcal{G}$ is cyclic, alternating, or of order at most $f(k)$.*

Still more generally Babai [2] showed that no class of finite graphs which is closed under subgraphs and contractions can be universal, except for the class of all graphs.

We could quantify Frucht's Theorem by asking: given a group $\mathcal{G}$, what is the smallest number of vertices of a graph G with $\mathrm{Aut}(G) = \mathcal{G}$? This function will behave in a very erratic fashion. For example, the symmetric group $\mathcal{S}_n$ is the automorphism group of the null graph on n vertices, but the smallest graph whose automorphism group is the alternating group $\mathcal{A}_n$ has about 2^n vertices (the exact number was calculated by Liebeck [41]). Clearly this number is not smaller than the degree of the smallest faithful permutation representation of $\mathcal{G}$; this has been investigated in detail by Babai *et al.* [6].

Other measures of the "size" of the graph could be used; for example, the number of edges, or the number of orbits of $\mathcal{G}$ on vertices or edges. See [5] for a survey of results on these questions.

Given that almost all graphs admit only the trivial group, we might wonder if perhaps Frucht's Theorem can be strengthened to state that almost all graphs which admit a given group $\mathcal{G}$ in fact have $\mathcal{G}$ as their full automorphism group. This holds for some but not all groups. Cameron [9] proved the following theorem:

Theorem 5.4 *Given a group $\mathcal{G}$, consider those n-vertex graphs whose automorphism group contains $\mathcal{G}$. The proportion of such graphs whose automorphism group is precisely $\mathcal{G}$ tends to a limit $a(\mathcal{G})$ as $n \to \infty$.*

The limit $a(\mathcal{G})$ is a rational number, but unlike the case where $\mathcal{G}$ is the trivial group, it is not necessarily 1. For example, if $\mathcal{G}$ is the dihedral group of order 10, then the limit is $1/3$. This is because, of those graphs which admit $\mathcal{G}$, almost all have $\mathcal{G}$ acting on a set of five vertices and fixing the rest of the graph. (A random graph admitting $\mathcal{G}$ consists of a random graph on $n-5$ vertices, and five 'special' vertices on which $\mathcal{G}$ acts as the symmetry group of a pentagon, all joined to the same random subset of the other $n-5$ vertices.) The induced subgraph on the five special vertices may be a complete graph, a null graph, or a 5-cycle; only in the third case is $\mathcal{G}$ the full group (almost surely).

In fact, $a(\mathcal{G}) = 1$ if and only if $\mathcal{G}$ is a direct product of symmetric groups. For abelian groups $\mathcal{G}$, $a(\mathcal{G}) = 1$ if $\mathcal{G}$ is an elementary abelian 2-group (possibly trivial), and $a(\mathcal{G}) = 0$ otherwise; but the values of $a(\mathcal{G})$ for metabelian groups are dense in the unit interval.

It would be interesting to know whether similar results hold under hypotheses tending to work against such very local symmetries. For example, does a similar result hold for regular graphs?

6 Cayley graphs

A permutation group $\mathcal{G}$ is *regular* if it is transitive and only the identity stabilises a point. Any regular action can be identified with the action of the group on itself by right multiplication, where the group element g induces the permutation $x \mapsto xg$ of $\mathcal{G}$. This is the action used by Cayley to show that every group is isomorphic to a permutation group.

The orbits on pairs of this group are parametrised by group elements: they have the form $O_g = \{(x, gx) : x \in \mathcal{G}\}$ for $g \in \mathcal{G}$. The orbit paired with O_g is $O_{g^{-1}}$. So our description of a $\mathcal{G}$-invariant graph specialises in the following way.

> Let S be a subset of $\mathcal{G}$, closed under taking inverses, and not containing the identity. The *Cayley graph* $\mathrm{Cay}(\mathcal{G}, S)$ has vertex set $\mathcal{G}$, and edges $\{x, sx\}$ for all $s \in S$, $x \in \mathcal{G}$.

Since $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(\mathcal{G}, S))$, this Cayley graph is necessarily vertex-transitive (see the next section). It is connected if and only if the set S generates $\mathcal{G}$. Readers are warned that many authors use a different convention: the action of the group is by left multiplication, and the edges have the form $\{x, xs\}$ for $s \in S$, $x \in \mathcal{G}$. (The difference is immaterial.)

A *graphical regular representation* or GRR of a group $\mathcal{G}$ is defined to be a graph for which the regular action of $\mathcal{G}$ is the full automorphism group, that is, a Cayley graph $\mathrm{Cay}(\mathcal{G}, S)$ with $\mathrm{Aut}(\mathrm{Cay}(\mathcal{G}, S)) = \mathcal{G}$. A considerable amount of effort went into the determination of groups which have GRRs. The problem was finally solved by Hetzel [35] in the soluble case and Godsil [25] in general. First note that abelian groups of exponent greater than 2 never have GRRs, since any Cayley graph for the abelian group $\mathcal{G}$ also admits the automorphism $g \mapsto g^{-1}$. A *generalised dicyclic group* $\mathcal{G}$ is a group having a cyclic subgroup $\mathcal{H}$ of index 2 and an element g of order 4 such that $g^{-1}hg = h^{-1}$ for all $h \in \mathcal{H}$. (The quaternion group of order 8 is an example.) The theorem is as follows:

Theorem 6.1 *Let $\mathcal{G}$ be a finite group. Then $\mathcal{G}$ has a GRR if and only if $\mathcal{G}$ is not an abelian group of exponent greater than* 2 *or a generalised dicyclic group and is not one of thirteen exceptional groups (with orders at most* 32*).*

What about *random Cayley graphs* for $\mathcal{G}$, obtained by including inverse pairs of non-identity elements in S with probability $\frac{1}{2}$? Babai and Godsil [4] conjectured that, except for the two infinite classes in the theorem, almost all Cayley graphs for $\mathcal{G}$ are GRRs (that is, the probability that a random Cayley graph is a GRR tends to 1 as $|\mathcal{G}| \to \infty$). They proved that this is true in some cases (for example, non-abelian nilpotent groups of odd order).

Random Cayley graphs have other useful properties; for example, they are often expanders (Alon and Roichman [1]).

7 Vertex-transitive graphs

A graph G is *vertex-transitive* if the automorphism group of G acts transitively on the vertex set of G.

Any vertex-transitive graph has a description as a *Schreier coset graph*, generalising the representation of a Cayley graph above: we replace group elements by cosets of a subgroup $\mathcal{H}$ of $\mathcal{G}$ as vertices of the graph, and for adjacency, in place of an inverse-closed set of elements we use an inverse-closed set of double cosets of $\mathcal{H}$ in $\mathcal{G}$. Sabidussi [47] used this representation to show that any vertex-transitive graph has a multiple which is a Cayley graph. (Here, a *multiple* of G is obtained by replacing each vertex by a coclique of size k, and each edge by all possible edges between the corresponding cocliques, for some k.)

Not every vertex-transitive graph is a Cayley graph.The smallest counterexample is, not surprisingly, the Petersen graph: it has no automorphism of order 2 which fixes no vertex. McKay and Praeger [44] have considered the class of non-Cayley vertex-transitive graphs.

Marušič and Jordan independently made the conjecture that any vertex-transitive graph has a group of automorphisms which acts *semiregularly* on vertices (that is, the stabiliser of any vertex is the identity, but the subgroup is not required to be transitive). This conjecture was extended by Klin, who conjectured that any 2-closed permutation group contains such a subgroup. The conjecture is still open, though Giudici [24] has made substantial progress on it recently.

Obviously, vertex-transitive graphs are regular. However, they have some special properties which are not shared by all regular graphs. From the work of Mader, Watkins, Little, Grant, Holton, Babai, and others, we take the following list. A graph is *vertex-primitive* if there is no equivalence relation on the vertex set preserved by all automorphisms, apart from the trivial relations (equality and the 'universal' equivalence).

Theorem 7.1 *Let G be a k-regular, connected, vertex-transitive graph of order n. Then* *(a) G is $\lfloor \frac{2}{3}(k+1) \rfloor$-connected (even k-connected if it is vertex-primitive), and is k-edge-connected;*

(b) G has a 1-factor if n is even;

(c) G has a cycle of length at least $\sqrt{6n}$;

(d) the product of the clique number and the independence number of G is at most n.

It has been conjectured that, with finitely many exceptions, a connected vertex-transitive graph is Hamiltonian. The Petersen graph, of course, is one of these exceptions: it has a Hamiltonian path but no Hamiltonian circuit. Only four exceptional graphs are currently known; all have Hamiltonian paths.

The *Hadwiger number* of a graph is the smallest k for which some component of the graph can be contracted to the complete graph K_k. Recently, Babai [3] and Thomassen [50] have obtained structure theorems for connected vertex-transitive graphs with prescribed Hadwiger number. A graph is *toroidal* if it is embeddable in the torus It is *ring-like* if the vertices can be partitioned into sets $S_0, \ldots, S_{n-1}$, such that all edges join vertices in the same set or in consecutive sets (mod n), and the automorphism group induces a cyclic or dihedral group on this family of sets.

Theorem 7.2 *A sufficiently large connected vertex-transitive graph with Hadwiger number k is either toroidal or ring-like (with the cardinalities of the sets S_i bounded by a function of the Hadwiger number in the ring-like case).*

Clearly, arbitrarily large toroidal vertex-transitive graphs can be obtained as quotients of plane lattices (for example, rectangular grids with opposite sides identified). The proof of this substantial result involves many geometrical ideas, including isoperimetric inequalities for the hyperbolic plane.

A related result of Thomassen [51] shows that there are only finitely many vertex-transitive graphs of given genus $g \geq 3$.

Two properties weaker than vertex-transitivity but stronger than regularity are walk-regularity and neighbourhood-regularity. The first of these is touched on in Chapter **??**; here we consider the second.

Let a graph H be given. A graph G is *locally H* if, for any vertex $v \in VG$, the induced subgraph on the set if neighbours of v is isomorphic to H. A graph is *neighbourhood-regular* if it is locally H for some H.

The problem of deciding whether, for given H, there is a graph which is locally H, is recursively unsolvable [8, 57]. Nevertheless, there are a number of positive results. For example:

- For various graphs H, all graphs which are locally H have been determined. See Hall [32] for locally Petersen graphs, for example.
- If H is regular and connected with girth at least 6, then there are infinite graphs which are locally H (Weetman [54]).
- If H is regular with diameter 2 and satifies some extra conditions, then every graph which is locally H is finite (Weetman [55]).

A property of graphs which does not obviously relate to symmetry but turns out to imply vertex-transitivity is compactness. The basic results on this concept are due to Tinhofer [52]. To define it, we note that any permutation g of $\{1, \ldots, n\}$ can be represented by a *permutation matrix* $P(g)$, and that $g \in \mathrm{Aut}(G)$ if and only if $P(g)$ commutes with A, that is, $AP(g) = P(g)A$, where A is the adjacency matrix of G.

A matrix M is *doubly stochastic* if its entries are non-negative and all its row and column sums are equal to 1. By Birkhoff's Theorem (see Chapter **??**, Theorem 4.2), any doubly stochastic matrix is a convex combination of permutation matrices.

The graph G, with adjacency matrix A, is said to be *compact* if every doubly stochastic matrix M which commutes with A is a convex combination of permutation matrices which commute with A, that is, automorphisms of A.

The set of doubly stochastic matrices which commute with A is a polytope. So, if G is compact, its automorphisms are precisely the extreme points of this polytope, and so they can be found efficiently by linear programming. Birkhoff's Theorem shows that the complete graph is compact.

The meaning of compactness for arbitrary graphs is somewhat mysterious. But for regular graphs, we have the following. Note that if a graph is compact then so is its complement.

Theorem 7.3 *Let G be a compact connected regular graph. Then any two vertices of G can be interchanged by an automorphism of G. In particular, G is vertex-transitive.*

The converse of this theorem is false. If G is compact and regular with valency k, then $(1/k)A$ is a doubly stochastic matrix commuting with A, and so is a convex combination of automorphisms. Each such automorphism g has the property that $vg \sim v$ for all vertices v. However, many vertex-transitive graphs (for example, the Petersen graph) have no non-identity automorphisms with this property.

For more on compact graphs, see Godsil [14]. A more general concept called *weak compactness* is considered by Evdokimov *et al.* [17].

8 Higher symmetry

Symmetry conditions related to (and mostly stronger than) vertex-transitivity have had a lot of attention, often using group-theoretic techniques. One of the simplest is edge-transitivity, which usually implies vertex-transitivity; indeed, an edge-transitive graph which is not vertex-transitive is bipartite, and the bipartite blocks are the orbits of the automorphism group. (The complete bipartite graph $K_{m,n}$) with $m \neq n$ is a simple example.) There are also graphs which are vertex-transitive and edge-transitive but not *arc-transitive*, where an arc is a directed edge (or ordered pair of adjacent vertices).

For arc-transitive graphs, the connection between graph and group becomes particularly strong. We refer to Chapter **??** for a survey.

A strengthening of arc-transitivity is *distance-transitivity*, where we require the automorphism group to act transitively on pairs of vertices at distance i for $i = 0, 1, \ldots, d$, where d is the diameter of the graph. A major research effort directed at the determination of all such graphs is discussed in Chapter **??**.

An even stronger symmetry condition is homogeneity: a graph G is *homogeneous* if any isomorphism between (finite) induced subgraphs extends to an automorphism of G. The finite homogeneous graphs were determined by Sheehan and Gardiner (see [22]):

Theorem 8.1 *A finite homogeneous graph is one of the following:*

- *a disjoint union of complete graphs of the same size;*
- *a regular complete multipartite graph;*
- *the* 5*-cycle* C_5*;*
- *the line graph of* $K_{3,3}$.

More generally, we say that a graph G is *t-homogeneous* if any isomorphism between induced subgraphs of size at most t extends to an automorphism of G. Now t-homogeneity obviously implies the combinatorial property $C(t)$ defined in Chapter **??**. The list of graphs which satisfy $C(5)$ is the same as the list of homogeneous graphs in the preceding theorem, so the hypothesis may be weakened to 5-homogeneity.

9 Infinite graphs

We turn now to infinite graphs. Here, there are two very different areas of research, the first for locally finite graphs (those in which every vertex has finite valency), and the second for general graphs (but usually requiring homogeneity or some model-theoretic notions).

For a locally finite graph, the notion of an end (introduced by Halin [30] in graph theory, though used earlier by group theorists) is crucial. A *ray* is a one-way-infinite path in a graph. König's Infinity Lemma [39] shows that any infinite, locally finite, connected graph contains a ray. Let $\mathcal{R}(G)$ be the set of rays in G. We define an equivalence relation $\equiv$ on $\mathcal{R}(G)$ by the rule that $R_1 \equiv R_2$ if there is a ray R_3 intersecting both R_1 and R_2 in infinitely many vertices. The equivalence classes of $\equiv$ are the *ends* of G. We denote the set of ends by $\mathcal{E}(G)$.

It can be shown that the number of ends of the locally finite graph G is the supremum of the number of infinite components of $G - S$, over all finite subsets S of VG. (This result does not distinguish among infinite cardinals: we just say that the supremum of an unbounded set of natural numbers is ∞.)

For example, the integer lattice graph $\mathbb{Z}^k$ has just one end for $k > 1$, and two ends for $k = 1$; an infinite tree with valency greater than 2 has infinitely many (indeed, uncountably many) ends.

The main results connecting ends and automorphisms are the following theorems due to Halin [31] and Jung [38] respectively:

Theorem 9.1 *Any automorphism of a connected infinite locally finite graph fixes either an end or a finite subgraph.*

Theorem 9.2 *Let G be infinite, locally finite, and connected, and suppose that* $\mathrm{Aut}(G)$ *has only finitely many orbits on VG. Then the number of ends of G is* 1, 2 *or* $2^{\aleph_0}$.

For a locally finite graph, we can also consider the rate of growth of the number a_n of vertices at distance at most n from a fixed vertex v. Of course, this number depends on the chosen vertex v; but, if the distance from u to v is d, then

$$a_{n-d}(v) \leq a_n(u) \leq a_{n+d}(v),$$

so the asymptotics of the rate of growth (for example, polynomial of degree k, exponential with constant c) do not depend on the chosen vertex, and we can talk of the *growth* of G. If $G = \mathrm{Cay}(\mathcal{G}, S)$, then the choice of finite generating set S for $\mathcal{G}$ also does not affect the asymptotics of growth, and we can talk of the growth of $\mathcal{G}$. (Note, however, that a group can act vertex-transitively on each of two graphs with different growth.)

The growth is polynomial of degree k for the integer lattice $\mathbb{Z}^k$, and exponential with constant $k-1$ for the k-valent tree.

This different behaviour is not unconnected with the number of ends:

Theorem 9.3 *Let G be an infinite, locally finite, connected graph whose automorphism group has only finitely many orbits.*

(a) If the growth of G is bounded by a polynomial, then it satisfies

$$c_1 n^k \leq a_n \leq c_2 n^k$$

where k is a positive integer and $c_1, c_2 > 0$.

(b) G has linear growth if and only if it has two ends.

(c) If G is a Cayley graph of $\mathcal{G}$, then the growth is polynomial if and only if $\mathcal{G}$ is nilpotent-by-finite.

(d) There exist groups whose growth is faster than polynomial but slower than exponential.

(e) If G has infinitely many ends, then it has exponential growth.

Here part (c) is the celebrated theorem of Wolf [58] and Gromov [27] on groups of polynomial growth; see Trofimov [53] for an extension to vertex-transitive graphs. Part (b) follows from Gromov and from Seifter and Trofimov [48]; and an example for part (d) is the *Grigorchuk group* [26]. For a survey of vertex-transitive graphs with polynomial growth, see [37]

There are connections between growth of a graph and harmonic analysis. Lubotzky [42] gives an account of this material.

Macpherson [45] determined the infinite locally finite distance-transitive graphs. For any integers $s, t > 1$, there is an infinite tree which is semiregular, with valencies s and t in the two bipartite blocks. Now the graph $M(s,t)$ has as its vertex set the bipartite block of valency s; two vertices are adjacent if they lie at distance 2 in the tree.

Theorem 9.4 *A locally finite infinite distance-transitive graph is isomorphic to $M(s,t)$ for some $s, t > 1$.*

No such result holds without local finiteness. Some examples are given in Cameron [12].

Turning to arbitrary infinite graphs, we describe first the paradoxical result of Erdős and Rényi [16]: there is (up to isomorphism) *a unique countable random graph*; that is, there is a graph R such that a countable random graph R is isomorphic to R with probability 1. Moreover, R is highly symmetric: indeed, it is homogeneous (as defined in the preceding section). So the typical asymmetry of finite graphs does not hold in the countably infinite! A survey of the remarkable graph R appears in [11].

Other aspects of the theory are also very different in the countably infinite case, largely as a result of the random graph R. For example, in a group $\mathcal{G}$, a *square-root set* is a set $\sqrt{g} = \{x \in \mathcal{G} : x^2 = g\}$; it is *non-principal* if $g \neq 1$. The hypotheses of the following theorem of Cameron and Johnson are very mild.

Theorem 9.5 *Let $\mathcal{G}$ be a countable group which is not the union of a finite number of translates of non-principal square root sets. Then almost all random Cayley graphs for $\mathcal{G}$ are isomorphic to R.*

The classification of homogeneous graphs was extended to the infinite by Lachlan and Woodrow [40]. A graph is *universal K_n-free* if it contains no complete graph of size n but embeds every K_n-free graph as an induced subgraph. There is a unique countable homogeneous universal K_n-free graph for all $n \geq 2$: these graphs were first constructed by Henson [34] but their existence and uniqueness follows from a general construction method due to Fraïssé [19]. We denote this unique graph by H_n.

Theorem 9.6 *A countable homogeneous graph is one of the following:*

- *A disjoint union of complete graphs of the same size, or its complement (a complete multipartite graph);*
- *Henson's graph H_n (for $n \geq 3$) or its complement;*
- *the random graph R.*

10 Graph homomorphisms

In the final sections we turn to graph homomorphisms. The endomorphisms of a graph G (or homomorphisms from G to G) form a semigroup; the theory of semigroups is much less well-developed algebraically than that of groups. Nevertheless, homomorphisms in general are more revealing of the graph structure than automorphisms, and the theory has developed in surprising directions. The material here is based on [29], for which we refer for more details and references.

We will consider only finite simple graphs in this section.

A *homomorphism* from a graph G to a graph H is a function f from VG to VH such that, if $u \sim v$ in G, then $uf \sim vf$ in H. Thus, an isomorphism is a bijective homomorphism whose inverse is also a homomorphism. An *endomorphism* of G is a homomorphism from G to G.

A proper vertex-colouring of G with r colours is a map from VG to $\{1, \ldots, r\}$ such that adjacent vertices have distinct images. In other words, it is a homomorphism from G to the complete graph K_r. (For example, every bipartite graph has a homomorphism onto K_2.) Thus, the existence and enumeration questions for homomorphisms from G to H generalise the chromatic number and chromatic polynomial of G. So we expect these questions to be hard! We write $G \to H$ if there is a homomorphism from G to H, and think of this as saying that "G has a H-colouring".

If $G \to H$ and $H \to G$, then we say that G and H are *homomorphically equivalent*, and write $G \leftrightarrow H$. We write the homomorphic equivalence class of G as $[G]$. The set of such equivalence classes is partially ordered by the rule that $[G] \preceq [H]$ if $G \to H$. Quite a lot is known about this partial order: it is a lattice order, and it is dense (that is, if $[G] \prec [H]$, then there exists K with $[G] \prec [K] \prec [H]$).

The *independence ratio* $i(G)$ of a graph G is the ratio of the size of the largest independent set in G to the number of vertices of G. The *odd girth* of G is the size of the shortest odd cycle in G. As usual, ω and χ denote clique number and chromatic number. The following result gives a sufficient condition for the existence of homomorphisms from G to H:

Theorem 10.1 *Suppose that $G \to H$. Then*

- *$\omega(G) \geq \omega(H)$ and $\chi(G) \leq \chi(H)$;*
- *the odd girth of G is at least as great of that of H;*
- *if H is vertex-transitive then $i(G) \geq i(H)$.*

A *retraction* of G is a homomorphism f from G onto an induced subgraph H of G such that the restriction of f to VH is the identity map. The subgraph H is called a *retract* of G. it is easy to see that, given any endomorphism of G, some power of it is a retraction. Any retraction of a connected graph can be expressed as the composition of a sequence of foldings, where a *folding* is a homomorphism which identifies a unique pair of vertices.

The result of Sabidussi [47] mentioned at the start of Section 7 shows that every vertex-transitive graph is a retract of a Cayley graph.

Retracts play an important role in topology, based in part on the fact that a retract H of G is an *isometric subgraph* (that is, the distance between two vertices of H is the same if calculated in H or in G).

A graph is a *core* if it has no non-trivial retraction; equivalently, if every endomorphism is an automorphism. If H is a retract of G and is itself a core, then we say that H is a *core of G*. Now the following holds:

Theorem 10.2 *(a) If $G \leftrightarrow H$, then any core of G is isomorphic to any core of H. In particular, all cores of G are isomorphic (and we can speak of* the core of *G).*

(b) The core of G is (up to isomorphism) the smallest graph in $[G]$.

(c) The core of a vertex-transitive graph is vertex-transitive.

Many cores are known, for example all circulants of prime order, and the *Kneser graph* $K(r,s)$ whose vertices are the s-subsets of $\{1,\ldots,r\}$ with $r > 2s$, two vertices adjacent if they are disjoint (so that the Petersen graph is $K(5,2)$). For a survey of cores and their properties, see [33].

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